

On several kinds of sums involving balancing and Lucas-balancing numbers

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Abstract

A balancing number x is a natural number such that $8x^2 + 1$ is a perfect square and $\sqrt{8x^2 + 1}$ is called a Lucas-balancing number. In this paper, we explore several finite and infinite sums involving balancing and Lucas-balancing numbers.

1 Introduction

A positive integer x is called *balancing number* if

$$1 + 2 + \cdots + (x - 1) = (x + 1) + \cdots + (y - 1) \quad (1)$$

holds for some integer $y \geq x + 2$. The problem of determining all balancing numbers leads to the Pell's equation $8x^2 + 1 = y^2$ whose solutions in x can be described by the recurrence $B_n = 6B_{n-1} - B_{n-2}$ ($n \geq 2$) with $B_0 = 0$ and $B_1 = 1$. One of the most general extensions of balancing numbers is the problem of finding x such that (1)

$$1^k + 2^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l, \quad (2)$$

holds for some natural number y , where the exponents k and l are given positive integers (see [2, 5]). In the work of Liptai et al . [22] effective and non-effective finiteness theorems on (2) are proved. In [18] a balancing problem of ordinary binomial coefficients is studied.

In addition, C_n denotes the n -th *Lucas-balancing number*, satisfying $C_n = 6C_{n-1} - C_{n-2}$ ($n \geq 2$) with $C_0 = 1$ and $C_1 = 3$.

2 Properties of balancing numbers

In this section, we shall give several basic properties of balancing and Lucas-balancing numbers. The Binet forms for B_n and C_n are given by

$$B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}} \quad \text{and} \quad C_n = \frac{\alpha^n + \beta^n}{2} \quad (n \geq 0),$$

where $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$.

The balancing and Lucas-balancing numbers satisfy the identities

$$B_{n-r}B_{n+r} = B_n^2 - B_r^2, \quad \text{and} \quad C_{n-r}C_{n+r} = C_n^2 + C_r^2 - 1 \quad (n \geq r), \quad (3)$$

respectively ([27]). In particular, by putting $r = 1$, we have

$$B_{n-1}B_{n+1} = B_n^2 - 1, \quad \text{and} \quad C_{n-1}C_{n+1} = C_n^2 + 8 \quad (n \geq 1), \quad (4)$$

The identity

$$B_1 + B_3 + \cdots + B_{2n-1} = B_n^2$$

gives

$$B_{2n-1} = B_n^2 - B_{n-1}^2.$$

In general, for $a, b \geq 0$ we have

$$B_{a+b+1} = B_{a+1}B_{b+1} - B_aB_b.$$

There are some relations between B_n and C_n including $C_n = \sqrt{8B_n^2 + 1}$ ([25]).

In [25], several basic relations, which are analogous of $\sin(m \pm n)$, $\cos(m \pm n)$.

Lemma 1. *If m and n are integers and $m \geq n$, then*

1. $B_{n\pm m} = B_n C_m \pm B_m C_n$
2. $C_{n\pm m} = C_n C_m \pm 8B_m B_n$

Using the Binet formulas, one can verify $B_{-n} = -B_n$ and $C_{-n} = C_n$. From Lemma 1, we can easily get the following relations, which are used in the next section.

Proposition 1. 1. $B_{n+m} - 2B_n C_m = B_{m-n}$

2. $C_{n+m} - 2C_n C_m = -C_{m-n}$

In [28], several interesting identities are given for balancing numbers B_n and Lucas-balancing numbers C_n . Some of them are listed below. As usual $\left(\frac{a}{b}\right)$ denotes the Legendre symbol.

Proposition 2. 1. Let m and n be positive integers. Then $\gcd(B_m, B_n) = B_{\gcd(m, n)}$.

2. If p is an odd prime, then $C_p \equiv 3 \pmod{p}$ and $B_p \equiv \left(\frac{p}{8}\right) \pmod{p}$.
3. For any positive integer m , $B_{2m} \equiv 0 \pmod{C_m}$, $B_{2m-1} \equiv 1 \pmod{C_m}$.

In the following two propositions, we use the Binet forms of Balancing and Lucas-balancing numbers.

Proposition 3. For $n \geq 0$,

$$\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} 3^k B_k = \begin{cases} 2^{\frac{3n}{2}} B_n & \text{if } n \text{ is even;} \\ 2^{\frac{3(n-1)}{2}} C_n & \text{if } n \text{ is odd} \end{cases}$$

and

$$\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} 3^k C_k = \begin{cases} 2^{\frac{3n}{2}} C_n & \text{if } n \text{ is even;} \\ 2^{\frac{3(n+1)}{2}} B_n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We prove the first part only as the second part can be proved in a similar manner. Notice that $3\alpha - 1 = 2\sqrt{2}\alpha$ and $3\beta - 1 = -2\sqrt{2}\beta$. When n

is even, the left-hand side is equal to

$$\begin{aligned}
\frac{1}{4\sqrt{2}} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} 3^k (\alpha^k - \beta^k) &= \frac{(3\alpha - 1)^n - (3\beta - 1)^n}{4\sqrt{2}} \\
&= \frac{(2\sqrt{2}\alpha)^n - (2\sqrt{2}\beta)^n}{4\sqrt{2}} \\
&= (2\sqrt{2})^n B_n.
\end{aligned}$$

When n is odd, the left-hand side is equal to

$$\begin{aligned}
\frac{1}{4\sqrt{2}} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} 3^k (\alpha^k - \beta^k) &= \frac{(3\alpha - 1)^n - (3\beta - 1)^n}{4\sqrt{2}} \\
&= \frac{(2\sqrt{2}\alpha)^n + (2\sqrt{2}\beta)^n}{4\sqrt{2}} \\
&= \frac{(2\sqrt{2})^n C_n}{2\sqrt{2}}.
\end{aligned}$$

□

Proposition 4. For $n \geq 0$,

$$\sum_{k=0}^{2n} \binom{2n}{k} B_k = 8^n B_n, \tag{5}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} C_k = 8^n C_n, \tag{6}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k B_k = 4^n B_n, \tag{7}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k C_k = 4^n C_n. \tag{8}$$

Proof. Since, $(\alpha+1)^2 = 8\alpha$ and $(\beta+1)^2 = 8\beta$, we get (5) and (6), respectively. Further, (7) and (8) follows from $(\alpha-1)^2 = 4\alpha$ and $(\beta-1)^2 = 4\beta$ respectively.

□

3 The sums of products of two balancing numbers

In this section, we shall consider the sums of products of two balancing numbers. The sums of products of various numbers with or without binomial coefficients have been studied in ([1, 4, 7, 8, 11, 12, 14, 13, 14, 15, 16, 17, 30]). One of the famous results are about Bernoulli numbers \mathfrak{B}_n , defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \mathfrak{B}_n \frac{t^n}{n!}.$$

The following identity (with binomial coefficients) on sums of products of a pair of Bernoulli numbers is known as Euler's formula:

$$\sum_{i=0}^n \binom{n}{i} \mathfrak{B}_i \mathfrak{B}_{n-i} = -\mathfrak{B}_{n-1} - (n-1)\mathfrak{B}_n \quad (n \geq 0).$$

The new results (without binomial coefficients) are based upon the generating function of balancing and Lucas-balancing numbers of the form B_{kn+r} and C_{kn+r} , obtained in (11) and (13) respectively. Notice that the corresponding generating functions of Fibonacci numbers and Lucas numbers are given by

$$\frac{F_r + (-1)^r F_{k-r}t}{1 - L_k t + (-1)^k t^2} = \sum_{n=0}^{\infty} F_{kn+r} t^n \quad (9)$$

([3, (9)], [19, p.230] and

$$\frac{L_r + (-1)^r L_{k-r}t}{1 - L_k t + (-1)^k t^2} = \sum_{n=0}^{\infty} L_{kn+r} t^n \quad (10)$$

([3, (10)]), respectively.

Theorem 1. *Let k and r be fixed integers with $k > r \geq 0$. Then*

$$\begin{aligned} \sum_{m=0}^n B_{km+r} B_{k(n-m)+r} &= B_{k-r} \left(-(n+1) B_{k(n+1)+r} \right. \\ &\quad \left. + \sum_{j=0}^n \frac{B_k B_{k-r}^j}{2} \left(\frac{(-1)^j}{(B_k + B_r)^{j+1}} + \frac{1}{(B_k - B_r)^{j+1}} \right) (n-j+1) B_{k(n-j+1)+r} \right) \end{aligned}$$

and

$$\begin{aligned} \sum_{m=0}^n C_{km+r} C_{k(n-m)+r} &= C_{k-r} \left((n+1) C_{k(n+1)+r} \right. \\ &\quad - \sum_{j=0}^n \frac{2\sqrt{2}B_k(C_{k-r}\sqrt{-1})^j}{2} \left(\frac{1}{(2\sqrt{2}B_k + C_r\sqrt{-1})^{j+1}} \right. \\ &\quad \left. \left. + \frac{(-1)^j}{(2\sqrt{2}B_k - C_r\sqrt{-1})^{j+1}} \right) (n-j+1) C_{k(n-j+1)+r} \right). \end{aligned}$$

Proof. From the first identity of Proposition 1, we get $B_{mk+r} - 2C_k B_{(m-1)k+r} = -B_{(m-2)k+r}$ and $B_{k+r} - 2C_k B_r = B_{k-r}$. Hence, the generating function of B_{kn+r} is given by

$$b(t) := \frac{B_r + B_{k-r}t}{1 - 2C_k t + t^2} = \sum_{n=0}^{\infty} B_{kn+r} t^n. \quad (11)$$

Thus, we have

$$\begin{aligned} b(t)^2 &= \frac{(B_r + B_{k-r}t)^2}{B_{k+r} - 2B_r t - B_{k-r}t^2} b'(t) \\ &= \left(-B_{k-r} + \frac{B_r^2 + B_{k+r}B_{k-r}}{B_{k+r} - 2B_r t - B_{k-r}t^2} \right) b'(t) \\ &= \left(-B_{k-r} + \frac{1}{B_{k-r}} \frac{B_k^2}{\left(\frac{B_k}{B_{k-r}}\right)^2 - \left(\frac{B_r}{B_{k-r}} + t\right)^2} \right) b'(t) \\ &= B_{k-r} \left(-1 + \sum_{l=0}^{\infty} \left(\frac{B_r}{B_k} + \frac{B_{k-r}}{B_k} t \right)^{2l} \right) b'(t) \\ &= B_{k-r} \left(-1 + \sum_{l=0}^{\infty} \sum_{j=0}^{2l} \binom{2l}{j} \left(\frac{B_r}{B_k} \right)^{2l-j} \left(\frac{B_{k-r}}{B_k} \right)^j t^j \right) b'(t). \end{aligned}$$

Since

$$b'(t) = \sum_{n=0}^{\infty} (n+1) B_{k(n+1)+r} t^n,$$

we get

$$\begin{aligned}
b(t)^2 &= B_{k-r} \left(- \sum_{n=0}^{\infty} (n+1) B_{k(n+1)+r} t^n \right. \\
&\quad \left. + \sum_{l=0}^{\infty} \sum_{j=0}^{2l} \binom{2l}{j} \left(\frac{B_r}{B_k} \right)^{2l-j} \left(\frac{B_{k-r}}{B_k} \right)^j \sum_{n=j}^{\infty} (n-j+1) B_{k(n-j+1)+r} t^n \right) \\
&= B_{k-r} \sum_{n=0}^{\infty} \left(-(n+1) B_{k(n+1)+r} \right. \\
&\quad \left. + \sum_{j=0}^{n+1} \sum_{l=\lfloor \frac{j+1}{2} \rfloor}^{\infty} \binom{2l}{j} \left(\frac{B_r}{B_k} \right)^{2l-j} \left(\frac{B_{k-r}}{B_k} \right)^j (n-j+1) B_{k(n-j+1)+r} \right) t^n.
\end{aligned} \tag{12}$$

Since for a positive integer s and a real number x with $|x| < 1$

$$\frac{(1+x)^{-s} + (1-x)^{-s}}{2} = \binom{s-1}{s-1} + \binom{s+1}{s-1} x^2 + \binom{s+3}{s-1} x^4 + \dots$$

and

$$\frac{-(1+x)^{-s} + (1-x)^{-s}}{2} = \binom{s}{s-1} x + \binom{s+2}{s-1} x^3 + \binom{s+4}{s-1} x^5 + \dots,$$

we get

$$\begin{aligned}
&\sum_{l=\lfloor \frac{j+1}{2} \rfloor}^{\infty} \binom{2l}{j} \left(\frac{B_r}{B_k} \right)^{2l-j} \left(\frac{B_{k-r}}{B_k} \right)^j \\
&= \frac{1}{2} \left((-1)^j \left(1 + \frac{B_r}{B_k} \right)^{-j-1} + \left(1 - \frac{B_r}{B_k} \right)^{-j-1} \right) \left(\frac{B_{k-r}}{B_k} \right)^j \\
&= \frac{B_k B_{k-r}^j}{2} \left(\frac{(-1)^j}{(B_k + B_r)^{j+1}} + \frac{1}{(B_k - B_r)^{j+1}} \right).
\end{aligned}$$

Since

$$b(t)^2 = \sum_{n=0}^{\infty} \sum_{m=0}^n B_{km+r} B_{k(n-m)+r},$$

by comparing the coefficients on both sides, we get the first identity.

The second identity of Proposition 1 gives $C_{mk+r} - 2C_k C_{(m-1)k+r} = -C_{(m-2)k+r}$ and $C_{k+r} - 2C_k C_r = -C_{k-r}$. Hence, the generating function of C_{kn+r} is given by

$$c(t) := \frac{C_r - C_{k-r}t}{1 - 2C_k t + t^2} = \sum_{n=0}^{\infty} C_{kn+r} t^n. \quad (13)$$

Similarly, by using the relation $\sqrt{C_k^2 - 1} = 2\sqrt{2}B_k$, we get

$$\begin{aligned} c(t)^2 &= \frac{(C_r - C_{k-r}t)^2}{C_{k+r} - 2C_r t + C_{k-r}t^2} c'(t) \\ &= \left(C_{k-r} + \frac{C_r^2 - C_{k+r}C_{k-r}}{C_{k+r} - 2C_r t + C_{k-r}t^2} \right) c'(t) \\ &= \left(C_{k-r} - \frac{1}{C_{k-r}} \frac{C_k^2 - 1}{\frac{C_k^2 - 1}{C_{k-r}^2} + \left(\frac{C_r}{C_{k-r}} - t \right)^2} \right) c'(t) \\ &= C_{k-r} \left(1 - \sum_{l=0}^{\infty} (-1)^l \left(\frac{C_r}{2\sqrt{2}B_k} - \frac{C_{k-r}}{2\sqrt{2}B_k} t \right)^{2l} \right) c'(t) \\ &= C_{k-r} \left(1 - \sum_{l=0}^{\infty} (-1)^l \sum_{j=0}^{2l} (-1)^j \binom{2l}{j} \left(\frac{C_r}{2\sqrt{2}B_k} \right)^{2l-j} \left(\frac{C_{k-r}}{2\sqrt{2}B_k} \right)^j t^j \right) c'(t). \end{aligned}$$

Since

$$c'(t) = \sum_{n=0}^{\infty} (n+1) C_{k(n+1)+r} t^n,$$

we obtain

$$\begin{aligned}
c(t)^2 &= C_{k-r} \left(\sum_{n=0}^{\infty} (n+1) C_{k(n+1)+r} t^n \right. \\
&\quad - \sum_{l=0}^{\infty} (-1)^l \sum_{j=0}^{2l} (-1)^j \binom{2l}{j} \left(\frac{C_r}{2\sqrt{2}B_k} \right)^{2l-j} \left(\frac{C_{k-r}}{2\sqrt{2}B_k} \right)^j \\
&\quad \times \sum_{n=j}^{\infty} (n-j+1) C_{k(n-j+1)+r} t^n \Big) \\
&= C_{k-r} \sum_{n=0}^{\infty} \left((n+1) C_{k(n+1)+r} \right. \\
&\quad - \sum_{j=0}^{n+1} (-1)^j \sum_{l=\lfloor \frac{j+1}{2} \rfloor}^{\infty} (-1)^l \binom{2l}{j} \left(\frac{C_r}{2\sqrt{2}B_k} \right)^{2l-j} \left(\frac{C_{k-r}}{2\sqrt{2}B_k} \right)^j \\
&\quad \times (n-j+1) C_{k(n-j+1)+r} \Big) t^n.
\end{aligned}$$

Since for a positive integer s and a real number x with $|x| < 1$

$$\frac{(1+x\sqrt{-1})^{-s} + (1-x\sqrt{-1})^{-s}}{2} = \binom{s-1}{s-1} - \binom{s+1}{s-1} x^2 + \binom{s+3}{s-1} x^4 - \dots$$

and

$$\begin{aligned}
&\frac{-(1+x\sqrt{-1})^{-s} + (1-x\sqrt{-1})^{-s}}{2} \\
&= \sqrt{-1} \left(\binom{s}{s-1} x - \binom{s+2}{s-1} x^3 + \binom{s+4}{s-1} x^5 - \dots \right),
\end{aligned}$$

we have

$$\begin{aligned}
& \sum_{l=\lfloor \frac{j+1}{2} \rfloor}^{\infty} (-1)^l \binom{2l}{j} \left(\frac{C_r}{2\sqrt{2}B_k} \right)^{2l-j} \left(\frac{C_{k-r}}{2\sqrt{2}B_k} \right)^j \\
&= \frac{1}{2} \left((-1)^j \left(1 + \frac{C_r}{2\sqrt{2}B_k} \right)^{-j-1} + \left(1 - \frac{C_r}{2\sqrt{2}B_k} \right)^{-j-1} \right) \left(\frac{C_{k-r}}{2\sqrt{2}B_k} \right)^j \\
&= \frac{2\sqrt{2}B_k(C_{k-r}\sqrt{-1})^j}{2} \\
&\quad \times \left(\frac{(-1)^j}{(2\sqrt{2}B_k + C_r\sqrt{-1})^{j+1}} + \frac{1}{(2\sqrt{2}B_k - C_r\sqrt{-1})^{j+1}} \right).
\end{aligned}$$

Since

$$c(t)^2 = \sum_{n=0}^{\infty} \sum_{m=0}^n C_{km+r} C_{k(n-m)+r},$$

by comparing the coefficients on both sides, we get the second identity. \square

The following Corollary is a consequence of the first identity of Theorem 1. It can be obtained replacing $r = 0$ in (12) and using the fact that $B_0 = 0$.

Corollary 1. *For a positive integer k , we have*

$$\sum_{m=0}^n B_{km} B_{k(n-m)} = B_k \sum_{l=1}^{\lfloor \frac{n+1}{2} \rfloor} (n - 2l + 1) B_{k(n-2l+1)}.$$

Using (9) and the Fibonacci identities

$$(-1)^r F_{k-r} + F_r L_k = F_{k+r} \quad \text{and} \quad (-1)^{k-r} F_{k+r} F_{k-r} + F_r^2 = (-1)^{k-r} F_k^2,$$

we can have the following similar identity for Fibonacci numbers.

Theorem 2. *Let k and r be fixed integers with $k > r \geq 0$. If $k - r$ is even, then*

$$\begin{aligned}
& \sum_{m=0}^n F_{km+r} F_{k(n-m)+r} = F_{k-r} \left(-(n+1) F_{k(n+1)+r} \right. \\
& \quad \left. + \sum_{j=0}^n (-1)^{rj} \frac{F_k F_{k-r}^j}{2} \left(\frac{(-1)^j}{(F_k + F_r)^{j+1}} + \frac{1}{(F_k - F_r)^{j+1}} \right) (n - j + 1) F_{k(n-j+1)+r} \right).
\end{aligned}$$

Further, if $k - r$ is odd, then

$$\begin{aligned} \sum_{m=0}^n F_{km+r} F_{k(n-m)+r} &= F_{k-r} \left((n+1) F_{k(n+1)+r} \right. \\ &\quad - \sum_{j=0}^n (-1)^{rj} \frac{F_k (F_{k-r} \sqrt{-1})^j}{2} \left(\frac{1}{(F_k + F_r \sqrt{-1})^{j+1}} \right. \\ &\quad \left. \left. + \frac{(-1)^j}{(F_k - F_r \sqrt{-1})^{j+1}} \right) (n-j+1) F_{k(n-j+1)+r} \right). \end{aligned}$$

Using (10) along with the identities

$$(-1)^{r-1} L_{k-r} + L_k L_r = L_{k+r}$$

and

$$(-1)^{k-r} L_{k+r} L_{k-r} - L_r^2 = (-1)^{k-r} L_k^2 - 4(-1)^r = (-1)^{k-r} 5F_k^2,$$

we have a similar identity for Lucas numbers also.

Theorem 3. *Let k and r be fixed integers with $k > r \geq 0$. If $k - r$ is even, then*

$$\begin{aligned} \sum_{m=0}^n L_{km+r} L_{k(n-m)+r} &= L_{k-r} \left((n+1) L_{k(n+1)+r} \right. \\ &\quad - \sum_{j=0}^n (-1)^{(r+1)j} \frac{\sqrt{5} F_k (L_{k-r} \sqrt{-1})^j}{2} \left(\frac{(-1)^j}{(\sqrt{5} F_k + L_r \sqrt{-1})^{j+1}} \right. \\ &\quad \left. \left. + \frac{1}{(\sqrt{5} F_k - L_r \sqrt{-1})^{j+1}} \right) (n-j+1) L_{k(n-j+1)+r} \right). \end{aligned}$$

Further, if $k - r$ is odd, then

$$\begin{aligned} \sum_{m=0}^n L_{km+r} L_{k(n-m)+r} &= L_{k-r} \left(-(n+1) L_{k(n+1)+r} \right. \\ &\quad + \sum_{j=0}^n (-1)^{(r+1)j} \frac{\sqrt{5} F_k L_{k-r}^j}{2} \left(\frac{(-1)^j}{(\sqrt{5} F_k + L_r)^{j+1}} \right. \\ &\quad \left. \left. + \frac{1}{(\sqrt{5} F_k - L_r)^{j+1}} \right) (n-j+1) L_{k(n-j+1)+r} \right). \end{aligned}$$

4 Reciprocal sums of balancing numbers

In this section, we study several results for reciprocal sums of Balancing numbers and Lucas-balancing numbers. Here $[x]$ denotes the integer part of a real number of x .

Many authors studied bounds for reciprocal sums involving terms from Fibonacci and other related numbers (e.g., see [9, 10, 20, 23, 29]). In [6], several identities are shown for generalized Fibonacci numbers G_n defined by

$$G_n = aG_{n-1} + G_{n-2} \quad (n \geq 2), \quad G_0 = 0, \quad G_1 = 1,$$

where a is a positive integer. Some of them are the following. Observe that if $a = 1$, then $G_n = F_n$.

Proposition 5.

$$\begin{aligned} (1) \quad & \left[\left(\sum_{k=n}^{\infty} \frac{1}{G_k} \right)^{-1} \right] = \begin{cases} G_n - G_{n-1} & \text{if } n \text{ is even and } n \geq 2; \\ G_n - G_{n-1} - 1 & \text{if } n \text{ is odd and } n \geq 1. \end{cases} \\ (2) \quad & \left[\left(\sum_{k=n}^{\infty} \frac{1}{G_k^2} \right)^{-1} \right] = \begin{cases} aG_{n-1}G_n - 1 & \text{if } n \text{ is even and } n \geq 2; \\ aG_{n-1}G_n & \text{if } n \text{ is odd and } n \geq 1. \end{cases} \\ (3) \quad & \left[\left(\sum_{k=n}^{\infty} \frac{1}{G_{2k}} \right)^{-1} \right] = G_{2n} - G_{2n-2} - 1 \quad (n \geq 1) \\ (4) \quad & \left[\left(\sum_{k=n}^{\infty} \frac{1}{G_{2k-1}} \right)^{-1} \right] = G_{2n-1} - G_{2n-3} \quad (n \geq 2) \end{aligned}$$

We shall show several analogous results about alternating sums of reciprocal balancing numbers B_n and Lucas-balancing numbers C_n .

Theorem 4. *Let l be a fixed positive number. Then for $n \geq 1$ we have*

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{B_{lk}} \right)^{-1} \right] = B_{ln} - B_{l(n-1)} - 1, \quad (14)$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{1}{C_{lk}} \right)^{-1} \right] = C_{ln} - C_{l(n-1)}. \quad (15)$$

Proof. By (3), we have $B_{ln}^2 = B_{l(n-1)}B_{l(n+1)} + B_l^2$. Since $B_{l(n+1)} - B_l^2 > B_{l(n-1)} + 1$ ($n \geq 1$), we have

$$\begin{aligned}
\frac{1}{B_{ln} - B_{l(n-1)} - 1} - \frac{1}{B_{ln}} &= \frac{1}{B_{ln} \left(\frac{B_{ln}}{B_{l(n-1)}+1} - 1 \right)} \\
&= \frac{1}{\frac{B_{l(n+1)}B_{l(n-1)}+B_l^2}{B_{l(n-1)}+1} - B_{ln}} \\
&= \frac{1}{B_{l(n+1)} - B_{ln} - \frac{B_{l(n+1)}-B_l^2}{B_{l(n-1)}+1}} \\
&> \frac{1}{B_{l(n+1)} - B_{ln} - 1}.
\end{aligned}$$

Thus, we have

$$\frac{1}{B_{ln} - B_{l(n-1)} - 1} > \sum_{k=n}^{\infty} \frac{1}{B_{lk}}. \quad (16)$$

On the other hand,

$$\begin{aligned}
\frac{1}{B_{ln} - B_{l(n-1)}} - \frac{1}{B_{ln}} &= \frac{1}{B_{ln} \left(\frac{B_{ln}}{B_{l(n-1)}} - 1 \right)} \\
&= \frac{1}{\frac{B_{l(n+1)}B_{l(n-1)}+B_l^2}{B_{l(n-1)}} - B_{ln}} \\
&= \frac{1}{B_{l(n+1)} - B_{ln} + \frac{B_l^2}{B_{l(n-1)}}} \\
&< \frac{1}{B_{l(n+1)} - B_{ln}}.
\end{aligned}$$

Hence, we conclude that

$$\sum_{k=n}^{\infty} \frac{1}{B_{lk}} > \frac{1}{B_{ln} - B_{l(n-1)}}. \quad (17)$$

Combining (16) and (17), we obtain the identity (14). Similarly, using $C_{ln}^2 = C_{l(n-1)}C_{l(n+1)} - C_l^2 + 1$, we obtain the identity (4). \square

Theorem 5.

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{B_k} \right)^{-1} \right] = \begin{cases} B_n + B_{n-1} & \text{if } n \text{ is even;} \\ -(B_n + B_{n-1} + 1) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Firstly, we prove that

$$\frac{1}{B_{n+2} + B_{n+1}} + \frac{1}{B_n} - \frac{1}{B_{n+1}} < \frac{1}{B_n + B_{n-1}}. \quad (18)$$

Since $B_n^2 - B_{n-1}B_{n+1} = 1$, we have

$$\begin{aligned} & \left(\frac{1}{B_{n+1}} - \frac{1}{B_{n+2} + B_{n+1}} \right) - \left(\frac{1}{B_n} - \frac{1}{B_n + B_{n-1}} \right) \\ &= \frac{B_{n+2}}{B_{n+1}(B_{n+2} + B_{n+1})} - \frac{B_{n-1}}{B_n(B_n + B_{n-1})} \\ &= \frac{1}{B_{n+1} \left(1 + \frac{B_n}{B_{n+1}} + \frac{1}{B_{n+2}B_{n+1}} \right)} - \frac{1}{B_n \left(\frac{B_{n+1}}{B_n} + \frac{1}{B_n B_{n-1}} + 1 \right)} \\ &= \frac{1}{B_{n+1} + B_n + \frac{1}{B_{n+2}}} - \frac{1}{B_{n+1} + B_n + \frac{1}{B_{n-1}}} \\ &> 0. \end{aligned}$$

Therefore, repeating (18), we obtain

$$\frac{1}{B_n + B_{n-1}} > \sum_{i=0}^{\infty} \left(\frac{1}{B_{n+2i}} - \frac{1}{B_{n+2i+1}} \right). \quad (19)$$

Next, we show that

$$\frac{1}{B_n + B_{n-1} + 1} < \frac{1}{B_n} - \frac{1}{B_{n+1}} + \frac{1}{B_{n+2} + B_{n+1} + 1}. \quad (20)$$

Because of the inequality

$$\frac{B_{n+1} - 1}{B_{n-1} + 1} > 1 > \frac{B_n - 1}{B_{n+2} + 1},$$

we find

$$\begin{aligned}
& \left(\frac{1}{B_n} - \frac{1}{B_n + B_{n-1} + 1} \right) - \left(\frac{1}{B_{n+1}} - \frac{1}{B_{n+2} + B_{n+1} + 1} \right) \\
&= \frac{1}{B_n \left(\frac{B_n}{B_{n-1}+1} + 1 \right)} - \frac{1}{B_{n+1} \left(\frac{B_{n+1}}{B_{n+2}+1} + 1 \right)} \\
&= \frac{1}{\frac{B_{n-1}B_{n+1}+1}{B_{n-1}+1} + B_n} - \frac{1}{\frac{B_{n+2}B_{n+1}}{B_{n+2}+1} + B_{n+1}} \\
&= \frac{1}{B_{n+1} + B_n - \frac{B_{n+1}-1}{B_{n-1}+1}} - \frac{1}{B_{n+1} + B_n - \frac{B_{n+1}-1}{B_{n+2}+1}} \\
&> 0.
\end{aligned}$$

Repeating (20), we obtain

$$\frac{1}{B_n + B_{n-1} + 1} < \sum_{i=0}^{\infty} \left(\frac{1}{B_{n+2i}} - \frac{1}{B_{n+2i+1}} \right). \quad (21)$$

Combining (19) and (21), we get

$$\frac{1}{B_n + B_{n-1} + 1} < \sum_{k=n}^{\infty} \frac{(-1)^k}{B_k} < \frac{1}{B_n + B_{n-1}}$$

if n is even, and

$$-\frac{1}{B_n + B_{n-1} + 1} > \sum_{k=n}^{\infty} \frac{(-1)^k}{B_k} > -\frac{1}{B_n + B_{n-1}}$$

if n is odd. □

Theorem 6.

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{B_k^2} \right)^{-1} \right] = \begin{cases} B_n^2 + B_{n-1}^2 & \text{if } n \text{ is even;} \\ -(B_n^2 + B_{n-1}^2 + 1) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Our first objective is to show that

$$\frac{1}{B_{n+2}^2 + B_{n+1}^2} + \frac{1}{B_n^2} - \frac{1}{B_{n+1}^2} < \frac{1}{B_n^2 + B_{n-1}^2}. \quad (22)$$

Using $B_n^2 - B_{n-1}B_{n+1} = 1$, we have

$$\begin{aligned}
& \left(\frac{1}{B_{n+1}^2} - \frac{1}{B_{n+2}^2 + B_{n+1}^2} \right) - \left(\frac{1}{B_n^2} - \frac{1}{B_n^2 + B_{n-1}^2} \right) \\
&= \frac{B_{n+2}^2}{B_{n+1}^2(B_{n+2}^2 + B_{n+1}^2)} - \frac{B_{n-1}^2}{B_n^2(B_n^2 + B_{n-1}^2)} \\
&= \frac{1}{B_{n+1}^2 + \frac{(B_{n+2}B_{n+1})^2}{B_{n+2}^2}} - \frac{1}{B_n^2 + \frac{(B_{n+1}B_{n-1})^2}{B_{n-1}^2}} \\
&= \frac{1}{B_{n+1}^2 + B_n^2 + \frac{2B_n}{B_{n+2}} + \frac{1}{B_{n+2}^2}} - \frac{1}{B_{n+1}^2 + B_n^2 + \frac{2B_{n+1}}{B_{n-1}} + \frac{1}{B_{n-1}^2}} \\
&> 0,
\end{aligned}$$

from which (22) follows. Repeating (22), we obtain

$$\frac{1}{B_n^2 + B_{n-1}^2} > \sum_{i=0}^{\infty} \left(\frac{1}{B_{n+2i}^2} - \frac{1}{B_{n+2i+1}^2} \right). \quad (23)$$

Next, we prove that

$$\frac{1}{B_n^2 + B_{n-1}^2 + 1} < \frac{1}{B_n^2} - \frac{1}{B_{n+1}^2} + \frac{1}{B_{n+2}^2 + B_{n+1}^2 + 1}. \quad (24)$$

Since

$$\frac{B_{n+1}^2 - 2B_{n+1}B_{n-1} - 1}{B_{n-1}^2 + 1} > 0 > \frac{B_n^2 - 2B_{n+2}B_n - 1}{B_{n+2}^2 + 1},$$

we have

$$\begin{aligned}
& \left(\frac{1}{B_n^2} - \frac{1}{B_n^2 + B_{n-1}^2 + 1} \right) - \left(\frac{1}{B_{n+1}^2} - \frac{1}{B_{n+2}^2 + B_{n+1}^2 + 1} \right) \\
&= \frac{1}{B_n^2 \left(\frac{B_n^2}{B_{n-1}^2 + 1} + 1 \right)} - \frac{1}{B_{n+1}^2 \left(\frac{B_{n+1}^2}{B_{n+2}^2 + 1} + 1 \right)} \\
&= \frac{1}{\frac{(B_{n+1}B_{n-1}+1)^2}{B_{n-1}^2+1} + B_n^2} - \frac{1}{\frac{(B_{n+2}B_{n+1})^2}{B_{n+2}^2+1} + B_{n+1}^2} \\
&= \frac{1}{B_{n+1}^2 + B_n^2 - \frac{B_{n+1}^2 - 2B_{n+1}B_{n-1} - 1}{B_{n-1}^2 + 1}} - \frac{1}{B_{n+1}^2 + B_n^2 - \frac{B_n^2 - 2B_{n+2}B_n - 1}{B_{n+2}^2 + 1}} \\
&> 0.
\end{aligned}$$

Repeating (20), we obtain

$$\frac{1}{B_n^2 + B_{n-1}^2 + 1} < \sum_{i=0}^{\infty} \left(\frac{1}{B_{n+2i}^2} - \frac{1}{B_{n+2i+1}^2} \right). \quad (25)$$

Combining (23) and (25), we get

$$\frac{1}{B_n^2 + B_{n-1}^2 + 1} < \sum_{k=n}^{\infty} \frac{(-1)^k}{B_k^2} < \frac{1}{B_n^2 + B_{n-1}^2}$$

if n is even, and

$$-\frac{1}{B_n^2 + B_{n-1}^2 + 1} > \sum_{k=n}^{\infty} \frac{(-1)^k}{B_k^2} > -\frac{1}{B_n^2 + B_{n-1}^2}$$

if n is odd. □

Theorem 7.

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{B_{2k}} \right)^{-1} \right] = \begin{cases} B_{2n} + B_{2n-2} & \text{if } n \text{ is even;} \\ -(B_{2n} + B_{2n-2} + 1) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. To start with, we verify the inequality

$$\frac{1}{B_{2n}} - \frac{1}{B_{2n+2}} + \frac{1}{B_{2n+4} + B_{2n+2}} < \frac{1}{B_{2n} + B_{2n-2}}. \quad (26)$$

By virtue of the recurrence relation $B_n = 6B_{n-1} - B_{n-2}$, we have

$$\begin{aligned} B_n B_{n-4} - B_{n-1} B_{n-3} &= B_{n-1} B_{n-5} - B_{n-2} B_{n-4} \\ &= \dots \\ &= B_4 B_0 - B_3 B_1 = -35. \end{aligned} \quad (27)$$

Since

$$\begin{aligned} B_{2n+2} \left(\frac{B_{2n+2}}{B_{2n+4}} + 1 \right) &= B_{2n+2} + \frac{B_{2n+3} B_{2n+1} + 1}{B_{2n+4}} \\ &= B_{2n+2} + B_{2n} + \frac{-B_{2n+4} B_{2n} + B_{2n+3} B_{2n+1} + 1}{B_{2n+4}} \\ &= B_{2n+2} + B_{2n} + \frac{36}{B_{2n+4}} \end{aligned}$$

and

$$\begin{aligned}
B_{2n} \left(\frac{B_{2n}}{B_{2n-2}} + 1 \right) &= \frac{B_{2n+1}B_{2n-1} + 1}{B_{2n-2}} + B_{2n} \\
&= B_{2n+2} + B_{2n} + \frac{-B_{2n+2}B_{2n-2} + B_{2n+1}B_{2n-1} + 1}{B_{2n-2}} \\
&= B_{2n+2} + B_{2n} + \frac{36}{B_{2n-2}},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\left(\frac{1}{B_{2n+2}} - \frac{1}{B_{2n+4} + B_{2n+2}} \right) - \left(\frac{1}{B_{2n}} - \frac{1}{B_{2n} + B_{2n-2}} \right) \\
&= \frac{1}{B_{2n+2} \left(\frac{B_{2n+2}}{B_{2n+4}} + 1 \right)} - \frac{1}{B_{2n} \left(\frac{B_{2n}}{B_{2n-2}} + 1 \right)} \\
&= \frac{1}{B_{2n+2} + B_{2n} + \frac{36}{B_{2n+4}}} - \frac{1}{B_{2n+2} + B_{2n} + \frac{36}{B_{2n-2}}} \\
&> 0,
\end{aligned}$$

yielding (26).

Next, we show that

$$\frac{1}{B_{2n} + B_{2n-2} + 1} < \frac{1}{B_{2n}} - \frac{1}{B_{2n+2}} + \frac{1}{B_{2n+4} + B_{2n+2} + 1}. \quad (28)$$

Using (27), we get

$$\begin{aligned}
B_{2n} \left(\frac{B_{2n}}{B_{2n-2} + 1} + 1 \right) &= \frac{B_{2n+1}B_{2n-1} + 1}{B_{2n-2} + 1} + B_{2n} \\
&= B_{2n+2} + B_{2n} - \frac{B_{2n+2}B_{2n-2} - B_{2n+1}B_{2n-1} + B_{2n+2} - 1}{B_{2n-2} + 1} \\
&= B_{2n+2} + B_{2n} - \frac{B_{2n+2} - 36}{B_{2n-2} + 1}
\end{aligned}$$

and

$$\begin{aligned}
B_{2n+2} \left(\frac{B_{2n+2}}{B_{2n+4} + 1} + 1 \right) &= B_{2n+2} + \frac{B_{2n+3}B_{2n+1} + 1}{B_{2n+4} + 1} \\
&= B_{2n+2} + B_{2n} - \frac{B_{2n+4}B_{2n} - B_{2n+3}B_{2n+1} + B_{2n} - 1}{B_{2n+4} + 1} \\
&= B_{2n+2} + B_{2n} - \frac{B_{2n} - 36}{B_{2n+4} + 1}.
\end{aligned}$$

Since

$$\frac{B_{2n} - 36}{B_{2n+4} + 1} < \frac{B_{2n+2} - 36}{B_{2n-2} + 1},$$

we obtain

$$\begin{aligned} & \left(\frac{1}{B_{2n}} - \frac{1}{B_{2n} + B_{2n-2} + 1} \right) - \left(\frac{1}{B_{2n+2}} - \frac{1}{B_{2n+4} + B_{2n+2} + 1} \right) \\ &= \frac{1}{B_{2n} \left(\frac{B_{2n}}{B_{2n-2} + 1} + 1 \right)} - \frac{1}{B_{2n+2} \left(\frac{B_{2n+2}}{B_{2n+4} + 1} + 1 \right)} \\ &> 0, \end{aligned}$$

yielding (28). Repetition of (26) and (28) gives

$$\frac{1}{B_{2n} + B_{2n-2} + 1} < \sum_{k=n}^{\infty} \frac{(-1)^k}{B_{2k}} < \frac{1}{B_{2n} + B_{2n-2}}.$$

□

The following odd case can be proved similarly and hence it is omitted.

Theorem 8.

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{B_{2k+1}} \right)^{-1} \right] = \begin{cases} B_{2n+1} + B_{2n-1} & \text{if } n \text{ is even;} \\ -(B_{2n+1} + B_{2n-1} + 1) & \text{if } n \text{ is odd.} \end{cases}$$

The following result deals with the alternating sums of products of two consecutive balancing numbers.

Theorem 9.

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{B_k B_{k+1}} \right)^{-1} \right] = \begin{cases} B_n B_{n+1} + B_{n-1} B_n & \text{if } n \text{ is even;} \\ -(B_n B_{n+1} + B_{n-1} B_n + 1) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We first observe that

$$\begin{aligned} & \frac{1}{B_n B_{n+1}} - \frac{1}{B_{n+1} B_{n+2}} + \frac{1}{B_{n+2} B_{n+3} + B_{n+1} B_{n+2}} \\ & < \frac{1}{B_n B_{n+1} + B_{n-1} B_n}. \end{aligned} \quad (29)$$

By using $B_n^2 = B_{n-1}B_{n+1} + 1$, we get

$$\begin{aligned}
& \left(\frac{1}{B_{n+1}B_{n+2}} - \frac{1}{B_{n+2}B_{n+3} + B_{n+1}B_{n+2}} \right) \\
& - \left(\frac{1}{B_nB_{n+1}} - \frac{1}{B_nB_{n+1} + B_{n-1}B_n} \right) \\
& = \frac{1}{B_{n+1}B_{n+2} \left(1 + \frac{B_{n+1}}{B_{n+3}} \right)} - \frac{1}{B_nB_{n+1} \left(1 + \frac{B_{n+1}}{B_{n-1}} \right)} \\
& = \frac{1}{B_{n+1}B_{n+2} + (1 + B_nB_{n+2})\frac{B_{n+2}}{B_{n+3}}} - \frac{1}{B_nB_{n+1} + (1 + B_nB_{n+2})\frac{B_n}{B_{n-1}}} \\
& = \frac{1}{B_{n+1}B_{n+2} + B_nB_{n+1} + \frac{B_n+B_{n+2}}{B_{n+3}}} - \frac{1}{B_{n+1}B_{n+2} + B_nB_{n+1} + \frac{B_n+B_{n+2}}{B_{n-1}}} \\
& > 0
\end{aligned}$$

since

$$\frac{B_n + B_{n+2}}{B_{n+3}} < \frac{B_n + B_{n+2}}{B_{n-1}}.$$

We next show that

$$\begin{aligned}
& \frac{1}{B_nB_{n+1} + B_{n-1}B_n + 1} \\
& < \frac{1}{B_nB_{n+1}} - \frac{1}{B_{n+1}B_{n+2}} + \frac{1}{B_{n+2}B_{n+3} + B_{n+1}B_{n+2} + 1}. \quad (30)
\end{aligned}$$

In view of the inequality

$$\begin{aligned}
1 + B_{n-1}B_{n+1} + B_nB_{n+2} - B_{n+1}B_{n+2} & < 0 \\
& < 1 + B_nB_{n+2} + B_{n+1}B_{n+3} - B_nB_{n+1}.
\end{aligned}$$

we have

$$\begin{aligned}
& \left(\frac{1}{B_n B_{n+1}} - \frac{1}{B_n B_{n+1} + B_{n-1} B_n + 1} \right) \\
& - \left(\frac{1}{B_{n+1} B_{n+2}} - \frac{1}{B_{n+2} B_{n+3} + B_{n+1} B_{n+2} + 1} \right) \\
& = \frac{1}{B_n B_{n+1} \left(1 + \frac{B_n B_{n+1}}{B_{n-1} B_n + 1} \right)} - \frac{1}{B_{n+1} B_{n+2} \left(1 + \frac{B_{n+1} B_{n+2}}{B_{n+2} B_{n+3} + 1} \right)} \\
& = \frac{1}{B_n B_{n+1} + \frac{(1+B_{n-1} B_n + 1)(1+B_n B_{n+1})}{B_{n-1} B_n + 1}} \\
& - \frac{1}{B_{n+1} B_{n+2} + \frac{(1+B_n B_{n+1} + 1)(1+B_{n+1} B_{n+2})}{B_{n+2} B_{n+3} + 1}} \\
& = \frac{1}{B_n B_{n+1} + B_{n+1} B_{n+2} + \frac{1+B_{n-1} B_{n+1} + B_n B_{n+2} - B_{n+1} B_{n+2}}{B_{n-1} B_n + 1}} \\
& - \frac{1}{B_n B_{n+1} + B_{n+1} B_{n+2} + \frac{1+B_n B_{n+2} + B_{n+1} B_{n+3} - B_n B_{n+1}}{B_{n+2} B_{n+3} + 1}} \\
& > 0
\end{aligned}$$

By repeating (29) and (30), we get

$$\frac{1}{B_n B_{n+1} + B_{n-1} B_n + 1} < \sum_{k=n}^{\infty} \frac{(-1)^k}{B_k B_{k+1}} < \frac{1}{B_n B_{n+1} + B_{n-1} B_n}.$$

□

In a similar manner, we can prove the following two theorems.

Theorem 10.

$$\begin{aligned}
\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{B_{2k}^2} \right)^{-1} \right] &= \begin{cases} B_{2n}^2 + B_{2n-2}^2 & \text{if } n \text{ is even;} \\ -(B_{2n}^2 + B_{2n-2}^2 + 1) & \text{if } n \text{ is odd.} \end{cases} \\
\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{B_{2k-1}^2} \right)^{-1} \right] &= \begin{cases} B_{2n-1}^2 + B_{2n-3}^2 & \text{if } n \text{ is even;} \\ -(B_{2n-1}^2 + B_{2n-3}^2 + 1) & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{B_{2k-1}B_{2k+1}} \right)^{-1} \right] = \begin{cases} B_{2n}^2 + B_{2n-2}^2 - 1 & \text{if } n \text{ is even;} \\ -(B_{2n}^2 + B_{2n-2}^2) & \text{if } n \text{ is odd.} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{B_{2k}B_{2k+2}} \right)^{-1} \right] = \begin{cases} B_{2n+1}^2 + B_{2n-1}^2 - 1 & \text{if } n \text{ is even;} \\ -(B_{2n+1}^2 + B_{2n-1}^2) & \text{if } n \text{ is odd.} \end{cases}$$

For Lucas-balancing numbers C_n we have the following corresponding results.

Theorem 11.

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{C_k} \right)^{-1} \right] = \begin{cases} C_n + C_{n-1} - 1 & \text{if } n \text{ is even;} \\ -(C_n + C_{n-1}) & \text{if } n \text{ is odd.} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{C_k^2} \right)^{-1} \right] = \begin{cases} C_n^2 + C_{n-1}^2 - 1 & \text{if } n \text{ is even;} \\ -(C_n^2 + C_{n-1}^2) & \text{if } n \text{ is odd.} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{C_{2k}} \right)^{-1} \right] = \begin{cases} C_{2n} + C_{2n-2} - 1 & \text{if } n \text{ is even;} \\ -(C_{2n} + C_{2n-2}) & \text{if } n \text{ is odd.} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{C_{2k+1}} \right)^{-1} \right] = \begin{cases} C_{2n+1} + C_{2n-1} - 1 & \text{if } n \text{ is even;} \\ -(C_{2n+1} + C_{2n-1}) & \text{if } n \text{ is odd.} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{C_k C_{k+1}} \right)^{-1} \right] = \begin{cases} C_n C_{n+1} + C_{n-1} C_n - 1 & \text{if } n \text{ is even;} \\ -(C_n C_{n+1} + C_{n-1} C_n) & \text{if } n \text{ is odd.} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{C_{2k}^2} \right)^{-1} \right] = \begin{cases} C_{2n}^2 + C_{2n-2}^2 - 1 & \text{if } n \text{ is even;} \\ -(C_{2n}^2 + C_{2n-2}^2) & \text{if } n \text{ is odd.} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{C_{2k-1}^2} \right)^{-1} \right] = \begin{cases} C_{2n-1}^2 + C_{2n-3}^2 - 1 & \text{if } n \text{ is even;} \\ -(C_{2n-1}^2 + C_{2n-3}^2) & \text{if } n \text{ is odd.} \end{cases}$$

$$\left[\left(\sum_{k=n}^{\infty} \frac{(-1)^k}{C_{2k-1} C_{2k+1}} \right)^{-1} \right] = \begin{cases} C_{2n}^2 + C_{2n-2}^2 - 1 & \text{if } n \text{ is even;} \\ -(C_{2n}^2 + C_{2n-2}^2) & \text{if } n \text{ is odd.} \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{C_{2k} C_{2k+2}} \right)^{-1} \right\rfloor = \begin{cases} C_{2n+1}^2 + C_{2n-1}^2 - 1 & \text{if } n \text{ is even;} \\ -(C_{2n+1}^2 + C_{2n-1}^2) & \text{if } n \text{ is odd.} \end{cases}$$

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